THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Advanced Calculus I
Homework 1

Due Date: 11:59pm, 6 June, 2025

1. If $v(t)$ is smooth for any t such that $a \leq t \leq b$. Show how the arc-length formula

$$S = \int_a^b \|v'(t)\| dt$$

can be derived by approximating the curve $v(t)$ by straight line segments and taking a limit.

Solution. The idea is we approximate the curve $v(t)$ by straight segments. We pick points $t_0 = a, t_1, t_2, \ldots, t_{n-1}, t_n = b$ and connect these points by line segments $v(t_i) - v(t_{i-1})$ for $i = 1, \ldots, n$. Note that $v(t_i) - v(t_{i-1})$ is a vector. Then the length of the curve S is approximately

$$S \approx \sum_{i=1}^{n} \|v(t_i) - v(t_{i-1})\|.$$

Recall that the velocity vector at the point on the curve $v(t_i)$ is given by

$$v'(t_i) = \lim_{t \to t_i} \frac{v(t) - v(t_i)}{t - t_i}.$$

Then substituting this in to the expression above (the t_{i-1}'s are "close" to t_i and the approximate gets better as we take $n \to +\infty$) we continue to approximate

$$\begin{aligned} S &\approx \sum_{i=1}^{n} \|v(t_i) - v(t_{i-1})\| \\ &\approx \sum_{i=1}^{n} \|v'(t_i)\|(t_i - t_{i-1}). \end{aligned}$$

We recognize this last expression as a Riemann sum and hence taking $n \to +\infty$, we see that

$$S \approx \sum_{i=1}^{n} \|v'(t_i)\|(t_i - t_{i-1}) \to \int_a^b \|v'(t)\| dt$$

as required. ◀

2. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to [a, b]$ be continuous. Using the $\varepsilon - \delta$ definition of continuity, show that

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(g(a)).$$

Solution. Let $\varepsilon > 0$ be given. We want to show that there is a $\delta > 0$ such that for $0 < x - a < \delta$, $|f(g(x)) - f(g(a))| < \varepsilon$. Since f is continuous at $g(a)$, there is a $\delta_1 > 0$ such that for all $|y - g(a)| < \delta_1$, we have that $|f(y) - f(g(a))| < \varepsilon$. Now since g is continuous at a, there is a $\delta_2 > 0$ such that for all $0 < x - a < \delta_2$ (remember that x is in the interval $[a, b]$) such that $|g(x) - g(a)| < \delta_1$. Then taking $\delta = \delta_2$ and $y = g(x)$ yields the desired result. ◀

3. For a function f of one variable, construct a function such that
   (a) f is differentiable everywhere, and
   (b) f' is bounded everywhere, but
   (c) f' is not Riemann-integrable.

Solution. *Note: Since the solution for this question is more difficult, I was more generous in the grading of this question.*

The function that satisfies the conditions (a), (b), and (c) is called the Volterra's function, which we roughly explain below:

Let V be the Smith–Volterra–Cantor set, which is defined by an infinite process of removing intervals from the unit interval $[0, 1]$. We first remove the middle $1/4$ from the interval $[0, 1]$, so that what remains is

$$V_1 := \left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right].$$

Then we inductively remove subintervals of width $1/4^n$ from the middle of each of the 2^{n-1} remaining intervals. For example, at the second step we are left with

$$V_2 := \left[0, \frac{5}{32}\right] \cup \left[\frac{7}{32}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{25}{32}\right] \cup \left[\frac{27}{32}, 1\right].$$

The Smith–Volterra–Cantor set V is what remains of the interval $[0, 1]$ after infinitely many of these removals. By construction, V contains no intervals and therefore has empty interior, and is the intersection of closed sets and hence is closed. Intervals of total length

$$\sum_{n=0}^{\infty} \frac{2^n}{2^{2n+2}} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{2}$$

are removed from $[0, 1]$, which means that V consists only of boundary points and has a positive measure of $1/2$.

We now consider the function f defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then for $x \neq 0$, note that $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ which we observe oscillates between -1 and 1 infinitely fast at 0. We construct Volterra's function by placing countably many copies of f on the deleted intervals of the Smith–Volterra–Cantor set V. We first define f_1. Let x_0 be the largest value of x in the interval $\left[0, \frac{1}{8}\right]$ such that $f'(x_0) = 0$. Then we define $\tilde{f}_1$ on $\left[0, \dfrac{1}{8}\right]$ by

$$\tilde{f}_1(x) := \begin{cases} f(x), & 0 \leq x \leq x_0 \\ f(x_0), & x_0 \leq x \leq \frac{1}{8}. \end{cases}$$

We now extend $\tilde{f}_1$ to the interval $\left[0, \dfrac{1}{4}\right]$ by reflecting it across the vertical line $x = \dfrac{1}{8}$, i.e., we define $\hat{f}_1$ on $\left[0, \dfrac{1}{4}\right]$ by

$$\hat{f}_1(x) := \begin{cases} \tilde{f}_1(x), & 0 \leq x \leq \frac{1}{8} \\ \tilde{f}_1\left(-x + \frac{1}{4}\right), & \frac{1}{8} \leq x \leq \frac{1}{4}. \end{cases}$$

Note that since $f(0) = 0$, $\hat{f}_1(0) = 0$ and $\hat{f}_1\left(\frac{1}{4}\right) = 0$. Finally we define f_1 on $[0, 1]$ by translating $\hat{f}_1$ to the subinterval $\left[\frac{3}{8}, \frac{5}{8}\right]$ (the first subinterval removed in the first step of constructing V) and defining f_1 to be 0 outside this subinterval. That is,

$$f_1(x) := \begin{cases} 0, & 0 \leq x \leq \frac{3}{8} \\ \hat{f}_1\left(x - \frac{3}{8}\right), & \frac{3}{8} \leq x \leq \frac{5}{8} \\ 0, & \frac{5}{8} \leq x \leq 1. \end{cases}$$

Note that by construction, f_1 is 0 on V_1 and non-zero only on $[0, 1] \setminus V_1$ and f_1 oscillates infinitely fast between 1 and -1 at the points $x = \dfrac{3}{8}$ and $x = \dfrac{5}{8}$. We repeat repeat this construction on the subintervals $\left[\frac{5}{32}, \frac{7}{32}\right]$ and $\left[\frac{25}{32}, \frac{27}{32}\right]$ and add it to f_1 to produce f_2 and so on so that f_n is non-zero only on $[0, 1] \setminus V_n$ and oscillates infinitely fast between 1 and -1 on the boundary of V_n. Finally Volterra's function v is defined as the limit of this sequence of functions $f_1, f_2, \ldots, f_n, \ldots$.

Then by construction since f is differentiable everywhere v is differentiable everywhere with bounded derivative on $[0, 1]$. However, v' is discontinuous on the Smith–Volterra–Cantor set V (since it oscillates infinitely fast on the boundary of V and V equals its own boundary). Since v' is discontinuous on a set of positive measure, it is not Riemann integrable by the Lebesgue criterion for Riemann integrability, and we are done. ◀